

Extending Partial Orders to Dense Linear Orders

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1 Introduction

For any countable partially ordered set P , there is an injective embedding of P into a linear order P^* so that for all x and y in the domain of P , $x >_P y$ implies $x >_{P^*} y$. More strongly, each P can be injectively embedded in a linear P^* which is uniformly recursive in P . We can compute such a P^* from P by an effective recursion in which instances of comparability are added to those in P . In a typical step of the recursion, we have a pair $\langle x, y \rangle$ of distinct incomparable points and we extend the ordering so that $x > y$ by adding all instances of $x^* > y^*$ for which x^* is greater than or equal to x and y is greater than or equal to y^* .

Definition 1.1 Suppose that P is a countable partial ordering. We say that P^* is a *constrained η -extension* of P , if P^* is a dense linear order without endpoints, the domain of P^* is equal to the domain of P , and for all x and y in the domain of P , $x >_P y$ implies $x >_{P^*} y$.

J. Loś posed the following problem: Find a necessary and sufficient condition for a given countable partial ordering to have a constrained η -extension. Rutkowski [1995] obtained preliminary results, some necessary conditions and some sufficient conditions. We will give a logician's solution to Loś's question, by showing that it has no simple answer.

In the construction of a linear extension of P , we had to consider all pairs $\langle x, y \rangle$ from P and decide which of x or y would be above the other. In the

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construction of a constrained η -extension of P , we have to consider all pairs $\langle x, y \rangle$ and decide which is above the other, which z is above both, which z is below both, and which z is strictly between them. So, for each pair, there are infinitely many possible ways to satisfy the properties required of a constrained η -extension. If there were only finitely many possibilities, then we could appeal to the compactness theorem and conclude that if P has a constrained η -extension then it has one which is recursive in P' , the complete $\Sigma_1^0(P)$ set. But, the best upper bound that is available comes from the Kleene Basis Theorem: If P has a constrained η -extension, then it has one which is recursive in \mathcal{O}^P , the complete $\Pi_1^1(P)$ set.

We will show that P 's having a constrained η -extension is Σ_1^1 -complete property of P . Thus, the upper bound provided by the Kleene Basis Theorem is the best possible. Then, we will exhibit a natural Π_1^1 -norm on those countable partial orders without constrained η -extensions, and observe that this norm can be used to uniformly construct a constrained η -extension of P , whenever there is such an extension.

2 Π_1^1 sets and subtrees of ω^ω

In the following discussion, we will work with sequences of natural numbers. We use $\omega^{<\omega}$ to refer to the set of finite sequences from ω and use ω^ω to refer to the set of ω -sequences from ω . We will denote elements of $\omega^{<\omega}$ by lower case Greek letters such as σ and τ and denote elements of ω^ω by upper case Roman letters such as X and Y , we will write $\sigma \hat{\ } \tau$ for the concatenation of σ and τ , and we will write $\sigma \upharpoonright n$ and $X \upharpoonright n$ for the restrictions of σ and X to their first n elements, respectively. We will write $\langle k_0, k_2, \dots, k_{n-1} \rangle$ to refer to a sequence by explicitly specifying its elements, and will write $\sigma(n)$ and $X(n)$ to refer to the elements of σ and X with index n , respectively. We say that τ is an *immediate extension* of σ , if there is an n in ω such that $\tau = \sigma \hat{\ } \langle n \rangle$.

2.1 Almost-perfect trees

A *tree* is a subset of $\omega^{<\omega}$ such that whenever $\sigma \in T$ and $n \leq \text{length}(\sigma)$, then $\sigma \upharpoonright n \in T$. We order the elements of T by extension, $\sigma \leq_T \tau$ if and only if there is an n , less than or equal to the length of τ , such that $\sigma = \tau \upharpoonright n$. We let T_σ denote the set of sequences τ in T such that σ and τ are \leq_T -comparable.

If \mathcal{A} is a subset of ω^ω and Z is an element of ω^ω , then \mathcal{A} is $\Pi_1^1(Z)$ if there

is a recursive predicate $R(w, x, y, z)$ such that \mathcal{A} is defined by

$$X \in \mathcal{A} \quad \text{if and only if} \quad (\forall Y \in \omega^\omega)(\exists n \in \omega)R(n, X \upharpoonright n, Y \upharpoonright n, Z \upharpoonright n).$$

The above also specifies what is meant by a $\Pi_1^1(Z)$ subset of ω . When there is no parameter Z in the definition of \mathcal{A} , we say that \mathcal{A} is Π_1^1 .

We say that \mathcal{A} is Π_1^1 if there is a Z such that \mathcal{A} is $\Pi_1^1(Z)$. A Π_1^1 set \mathcal{A} is Π_1^1 -complete if for every Π_1^1 set \mathcal{B} there is a continuous function f such that for all X , $X \in \mathcal{B}$ if and only if $f(X) \in \mathcal{A}$. A set is Σ_1^1 -complete if its complement is Π_1^1 -complete.

Kleene [1955] gave a detailed analysis of the Π_1^1 sets, some of which we recall below. In particular, Kleene showed that the collection of (codes for) well-founded subtrees of $\omega^{<\omega}$ is a Π_1^1 set which is Π_1^1 -complete. In the following, we will need a technical refinement of Kleene's theorem.

Definition 2.1 We say that a subtree of $\omega^{<\omega}$ is *almost-perfect* if the following conditions hold.

- For all $\sigma \in T$, if σ has an immediate extension in T , then there is a k such that for all n greater than k , $\sigma \frown \langle n \rangle \in T$.
- If T_σ is not well-founded, then there are only finitely many k such that $\sigma \frown \langle k \rangle \in T$ and $T_{\sigma \frown \langle k \rangle}$ is well-founded.

Lemma 2.2 *There is a recursive function a , which maps subtrees of $\omega^{<\omega}$ to subtrees of $\omega^{<\omega}$, such that*

1. $a(T)$ is well-founded if and only if T is well-founded,
2. for each T in the domain of a , $a(T)$ is almost-perfect.

Proof: The proof that we give for Lemma 2.2 is based on a suggestion of S. Simpson. Our original proof used a detailed analysis of Kleene's \mathcal{O} . Simpson's substantially shorter proof is based on ideas of Marcone [1993].

We begin with some notation. For σ and τ in $\omega^{<\omega}$ define $\sigma \preceq \tau$ to indicate that σ and τ have equal length, and for all i in their domain $\sigma(i) \leq \tau(i)$.

Let T be a subtree of $\omega^{<\omega}$, and define $a(T)$ by

$$\tau \in a(T) \iff (\exists \sigma)[\sigma \in T \text{ and } \sigma \preceq \tau].$$

Since T is a subtree of $a(T)$, if T is not well-founded then so is $a(T)$. Conversely, if $a(T)$ is not well-founded then let X be its left-most path. If T were well-founded then set $\{\sigma : \sigma \in T \text{ and } (\exists i)(\sigma \preceq X \upharpoonright i)\}$ would be a

finitely branching tree with no infinite path. By König's lemma, this tree would be finite. But, then only finitely many elements of X would have a \preceq -predecessor in T , contradicting X 's being a path in $a(T)$. This establishes the first claim of the lemma.

Now, we check that $a(T)$ is almost-perfect. First, suppose that τ has an immediate extension $\tau \hat{\ } \langle k \rangle$ in $a(T)$, and let σ be an element of T such that $\sigma \preceq \tau \hat{\ } \langle k \rangle$. Then, for all n greater than or equal to k , $\sigma \preceq \tau \hat{\ } \langle k \rangle \preceq \tau \hat{\ } \langle n \rangle$. Consequently, for all n greater than or equal to k , $\tau \hat{\ } \langle n \rangle \in a(T)$.

Second, suppose that $\tau \in a(T)$, τ has length l_τ , and $a(T)_\tau$ is not well-founded. Let X denote the left-most path in $a(T)_\tau$. Since $a(T)$ is closed upwards under \preceq , for each ρ , if ρ has length l and $X \upharpoonright l \preceq \rho$, then $\rho \in a(T)$. Thus, for each n greater than $X(l_\tau)$, if Y is defined by the following,

$$Y(l) = \begin{cases} n, & \text{if } l = l_\tau; \\ X(l), & \text{otherwise.} \end{cases}$$

then Y is a path in $a(T)$. Consequently, for each n greater than $X(l_\tau)$, the $l_\tau + 1$ st entry in X , $a(T)_{\tau \hat{\ } \langle n \rangle}$ is not well-founded, and therefore τ has only finitely many immediate extensions $\tau \hat{\ } \langle k \rangle$ in $a(T)$ such that $a(T)_{\tau \hat{\ } \langle k \rangle}$ is well-founded. ■

Corollary 2.3 *The collection of nonempty almost-perfect well-founded trees is Π_1^1 -complete.*

3 The Σ_1^1 -completeness of \mathbf{L}

Definition 3.1 Suppose that $X \subseteq \mathbb{N}$. Let $\mathbf{L}(X)$ be

$$\mathbf{L}(X) = \left\{ e : \begin{array}{l} \{e\}^X \text{ is a partial ordering of } \omega \text{ and} \\ \text{there is a constrained } \eta\text{-extension of } \{e\}^X \end{array} \right\}$$

and let $\mathbf{L} = \{ \langle e, X \rangle : e \in \mathbf{L}(X) \}$.

Theorem 3.2 *There is a recursive function f such that for each e and X ,*

$$\{e\}^X \text{ is a well-founded subtree of } \omega^{<\omega} \text{ if and only if } f(e) \notin \mathbf{L}(X).$$

Thus, \mathbf{L} is a complete Σ_1^1 -predicate.

Proof: We define f by describing a uniformly recursive procedure to convert representations of nonempty almost-perfect trees T into presentations of partially ordered sets $P(T)$. We will ensure the following implications.

1. If T is almost-perfect and well-founded, then $P(T)$ has no constrained η -extension.
2. If T is almost-perfect and not well-founded, then $P(T)$ has a constrained η -extension.

We can then conclude the Σ_1^1 -completeness of \mathbb{L} from Corollary 2.3.

We define a function $p(\sigma, T)$ of two variables. The first variable is over elements of $\omega^{<\omega}$ and the second variable is over subsets of $\omega^{<\omega}$. For each σ and each almost-perfect tree T , $p(\sigma, T)$ is a T -recursive enumeration of a partially ordered set. If $\langle \rangle \notin T$, then T is empty and $p(\langle \rangle, T)$ is a T -recursive enumeration of the empty partial order. In this case, we let $P(T)$ be a T -recursive presentation of \mathbb{Z} , and thereby ensure that $P(T)$ has no constrained η -extension. Otherwise, $p(\langle \rangle, T)$ is a T -recursive enumeration of an infinite partially ordered set, and we let $P(T)$ be a T -recursive presentation of the same partially ordered set. The split into cases is uniformly T -recursive, and in the latter case the T -recursive presentation of $p(\langle \rangle, T)$ can be obtained uniformly from its T -recursively enumerable presentation.

We define $p(\langle \rangle, T)$ to be the limit of the following recursion on $\omega^{<\omega}$, starting from the empty sequence $\langle \rangle$, working through T , and terminating at nodes which do not belong to T . We speak of p as being defined by recursion, since in defining $p(\sigma, T)$ we may be called to evaluate all of the $p(\sigma \hat{\ } \langle k \rangle, T)$. More formally, p is defined as a fixed point for the following description, where the existence of such a fixed point is ensured by the Kleene fixed point theorem.

Suppose that T is an almost-perfect subtree of $\omega^{<\omega}$.

Terminating nodes. If $\sigma \notin T$ then, $p(\sigma, T)$ is the empty partial order.

Recursion nodes. If $\sigma \in T$, then we enumerate $p(\sigma, T)$ so as to produce the partial order depicted in Figure 1. We omit the stage-by-stage description of the enumeration of $p(\sigma, T)$ and describe the ordering itself.

The first component $R(\sigma, T)$ of $p(\sigma, T)$ is a semi-infinite countable dense linear order with greatest element $r(\sigma, T)$, and no least element. The second component $Q(\sigma, T)$ of $p(\sigma, T)$ is a partial order formed by amalgamating an ω -sequence $Q_i(\sigma, T)$ of partial orders as follows. At the bottom, $Q_0(\sigma, T)$ is a semi-infinite countable dense linear order with greatest element $a(0, \sigma, T)$. For each i greater than or equal to 0, $Q_{2i+1}(\sigma, T)$ is equal to $p(\sigma \hat{\ } \langle i \rangle, T)$ and $Q_{2i+2}(\sigma, T)$ is countable dense linear order with least element $b(i, \sigma, T)$ and greatest element $a(i, \sigma, T)$.

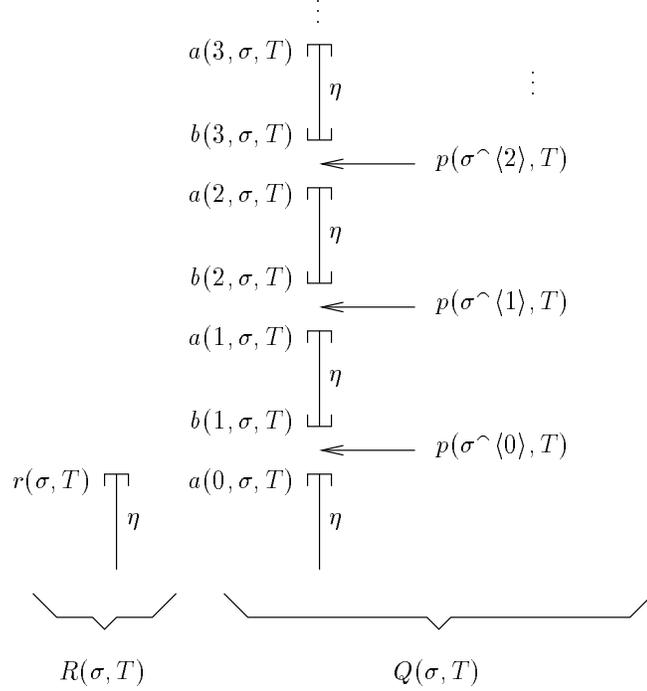


Figure 1: Defining $p(\sigma, T)$ by recursion.

For each x and y in $Q(\sigma, T) = \cup_{i \in \omega} Q_i(\sigma, T)$, $x < y$ if either there is a j such that $x \in Q_j(\sigma, T)$, $y \in Q_j(\sigma, T)$ and $x <_{Q_j(\sigma, T)} y$ or there are j and k such that $x \in Q_j(\sigma, T)$, $y \in Q_k(\sigma, T)$ and $j < k$.

Note that $p(\sigma, T)$'s being defined does not depend upon T_σ 's having no infinite path. If $\sigma \in T$, then $p(\sigma, T)$ will enumerate the elements of $R(\sigma, T)$ and all of the $Q_{2n}(\sigma, T)$ with the ordering described above. Additionally, the elements of the $Q_{2i+1}(\sigma, T)$ and their comparability with the elements in the other $Q_j(\sigma, T)$ are enumerated simultaneously with the enumeration of the $p(\sigma \hat{\ } \langle i \rangle, T)$.

It remains to show that T is not well-founded if and only if $P(T)$ has a constrained η -extension.

For the first half of this claim, we must show that if T is well-founded, then there is no constrained η -extension of $P(T)$. The claim is clear when T is empty, so assume that T is not empty and proceed by induction on the rank σ in T to show that $p(\sigma, T)$ has no constrained η -extension.

If σ has rank 0 in T , that is σ is a terminal node in T , then $p(\sigma, T)$ is the amalgamation of $R(\sigma, T)$, a linear order with a greatest element $r(\sigma, T)$, and $Q(\sigma, T)$, a linear order with a cofinal set of empty intervals $(a(i, \sigma, T), b(i+1, \sigma, T))$. These intervals are empty because every $p(\sigma \wedge \langle i \rangle, T)$ is empty. Suppose that P^* is a constrained extension of $p(\sigma, T)$ to a linear order. If in P^* , $r(\sigma, T)$ is above all of the $a(i, \sigma, T)$ then it is the greatest element of P^* and P^* is not a constrained η -extension of $p(\sigma, T)$. If, in P^* , $r(\sigma, T)$ is below one of the $a(i, \sigma, T)$, then all of $R(\sigma, T)$ lies below that $a(i, \sigma, T)$, $(a(i, \sigma, T), b(i+1, \sigma, T))$ is empty in P^* , and so P^* is not a constrained η -extension of $p(\sigma, T)$.

Now suppose that σ has rank greater than 0 in T . The partial orderings $p(\sigma \wedge \langle k \rangle, T)$ which are not empty are the results of our construction on nodes of smaller rank in T . By induction, none of the $p(\sigma \wedge \langle k \rangle, T)$ has a constrained η -extension. Now, the same argument as to the placement of $r(\sigma, T)$ used in the previous paragraph shows that $p(\sigma, T)$ cannot have a constrained η -extension.

For the second half of the claim, we suppose that T is not well-founded. We construct a constrained η -extension of T by an ω -length recursion. At step s , we consider all $\sigma \in \omega^{<\omega}$ of length s . For each such σ , either for all the elements x of $p(\sigma, T)$ and all the elements y of $P(T)$ we have already determined how x and y compare or we proceed as follows on σ .

By induction on s , we may assume that if we have not decided how the elements of $p(\sigma, T)$ compare to the other elements of $P(T)$ then T_σ is not well-founded. We choose $n(\sigma, T)$ so that for every m greater than or equal to $n(\sigma, T)$, $T_{\sigma \wedge \langle m \rangle}$ is not well-founded. Such a choice is possible, since T is almost-perfect. We choose a nonprincipal cut in $Q_{2n(\sigma, T)}(\sigma, T)$, and specify that each element of $R(\sigma, T)$ is below every upper bound of the cut and that $r(\sigma, T)$, the greatest element of $R(\sigma, T)$, is above every point which is a lower bound on the cut. That is to say that we insert $r(\sigma, T)$ into the cut. Let $Q(r(\sigma, T), \sigma, T)$ denote the points in $Q(\sigma, T)$ which are now below $r(\sigma, T)$. We extend the partial ordering of $Q(r(\sigma, T), \sigma, T)$ to a linear ordering, in an arbitrary way. We then embed $Q(r(\sigma, T), \sigma, T)$ into the set of nonprincipal cuts of the η -ordering $R(\sigma, T) \setminus \{r(\sigma, T)\}$ preserving their new order. Thus, we have made a constrained extension of $R(\sigma, T) \cup \bigcup_{i \leq 2n(\sigma, T)} Q_i(\sigma, T)$ and produced a dense linear ordering with greatest element. For i greater than $n(\sigma, T)$, we define our constrained extension on each $Q_{2i+1}(\sigma, T)$ during later steps in the recursion.

Note that if we do not completely specify P^* on all pairs with an element from $p(\sigma \wedge \langle i \rangle, T)$, then T_σ is not well-founded. Thus, we have ensured the validity of our induction assumption in the previous paragraph.

It is clear that the above procedure defines a partial order with no least and no greatest element. We now show that it is linear. During the initial step of the recursion, we extended the ordering of $P(T)$ so that for each x in $R(\langle \rangle, T)$ and each x in $Q(r(\langle \rangle, T), \langle \rangle, T)$, x becomes comparable to every other element of $P(T)$. The elements of the remaining $Q_{2i}(\langle \rangle, T)$ then have this property as well, since the only elements of $P(T)$ with which they were incomparable were those in $R(\langle \rangle, T)$. Then, we used recursion to linearize the remaining $Q_{2i+1}(\langle \rangle, T)$. Using induction and making the analogous observation, if $\sigma \in T$ then our recursion ensures that all of the elements of $R(\sigma, T)$ and of the $Q_{2i}(\sigma, T)$ are made to be comparable with all of the other elements of $P(T)$. But, for every element x of $P(T)$, there are σ and i such that either x is an element of $R(\sigma, T)$ or of $Q_{2i}(\sigma, T)$. Thus, every element of $P(T)$ is made to be comparable with every other one.

Finally, we show that our recursion produces a dense order. Let a and b be two elements of $P(T)$. There are only two cases in which there is not an x between a and b in $P(T)$. In the first case, there is a σ such that $a \in R(\sigma, T)$ and $b \in Q(\sigma, T)$ (or the symmetric case). But then, at some stage before or during that of the length of σ , $P(T)$ is extended so that a belongs to an η -interval, and so there is an interpolant between a and b in the extended order. In the second case, there are $\sigma \in T$ and i such that b is the greatest element of Q_{2i} , a is the least element of Q_{2i+2} , and $P(\sigma \hat{\ } \langle i \rangle)$ is empty because $\sigma \hat{\ } \langle i \rangle$ is not an element of T . Let s be the length of σ . Then, at some stage before or during s , $P(T)$ is extended so that a and b belong to the same η -interval. If it did not happen during an earlier stage, then i will be strictly less than $n(\sigma, T)$, the n chosen so that for every m greater than or equal to n , $T_{\sigma \hat{\ } \langle m \rangle}$ is not well-founded. But then, the ordering on $R(\sigma, T) \cup \bigcup_{j < 2n(\sigma, T)} Q_j$ is extended to be a dense linear ordering. In particular, both a and b belong to a dense interval in the extended partial ordering, and so there is an element between them in the extension. In all of the other possible configurations for a and b , either they belong to a dense interval in $P(T)$ or one belongs to a dense interval in $P(T)$, all of whose elements bear the same relation in \leq_T to the other, where these intervals are of type $R(\sigma, T)$ or $Q_{2i}(\sigma, T)$. In such a case, there is a point between them in $P(T)$, even before we make the extension.

Thus, we have verified the claim and proven the theorem. ■

Corollary 3.3 *L is not a Borel set.*

Proof: No $\mathbf{\Pi}_1^1$ -complete set is Borel. ■

We should also note that none of our reductions involved real parameters.

Consequently, the effective versions of Theorem 3.2 and Corollary 3.3 hold as well.

Theorem 3.4 *The set of indices for recursive partial orderings of ω which have constrained η -extensions is a Σ_1^1 -complete subset of ω . Consequently, this set is not Δ_1^1 .*

4 Finding constrained η -extensions

Definition 4.1 • A *norm* on a Π_1^1 set \mathcal{A} is a function φ from \mathcal{A} into the ordinals.

- Further, φ is a Π_1^1 -norm if φ is a norm and there are Π_1^1 and Σ_1^1 relations \leq_{φ}^{Π} and \leq_{φ}^{Σ} , respectively, such that for all Y in \mathcal{A} , for all X ,

$$\begin{aligned} [X \in \mathcal{A} \quad \text{and} \quad \varphi(X) \leq \varphi(Y)] &\iff X \leq_{\varphi}^{\Pi} Y \\ &\iff X \leq_{\varphi}^{\Sigma} Y. \end{aligned}$$

In other words, the initial segments of the domain of φ are uniformly Δ_1^1 .

Now, we define a Π_1^1 -norm associated with the existence of constrained η -extensions. Fix a recursive ω -ordering of $\omega \times \omega$. Suppose that P is a partial ordering of ω . Define the tree $R(P)$ so that the elements of $R(P)$ are sequences of finite linear orderings $\langle r_0, r_1, \dots, r_k \rangle$ on sets of natural numbers such that, for each i less than k the following conditions hold.

1. For each i less than k , r_i is a suborder of r_{i+1} .
2. There is a linear ordering of ω which extends both P and r_k . (More concretely, the axioms of a linear order are consistent with the collection of positive instances of comparability appearing in r_i or in P .)
3. For each i less than or equal to k , if $\langle n, m \rangle$ is the i th element of $\omega \times \omega$, then n and m belong to the domain of r_i , n and m are not maximal or minimal in r_i and there is an element of r_i which lies strictly between n and m in r_i .

Thus, $R(P)$ consists of sequences of finite constrained extensions of P such that the requirements of linearity, density, and unboundedness are met for the i th pair of integers within the i th element of the sequence. If $R(P)$ is not well-founded, then the direct limit of the partial orders in an infinite path

through $R(P)$ is a constrained η -extension of P . Conversely, any constrained η -extension of P can be used to produce an infinite path in $R(P)$.

Definition 4.2 Define ρ to map the complement of \mathbb{L} to ω_1 by letting $\rho(P)$ be the ordinal rank of $R(P)$.

Note that the map $P \mapsto R(P)$ is arithmetic.

Suppose that P has no constrained η -extension and that Q is a partial ordering of ω . First, $\rho(Q) < \rho(P)$ if and only if there is an order preserving embedding of $R(Q)$ into $R(P)$. This embedding is obtained by mapping a nodes in $R(Q)$ to nodes in $R(P)$ without decreasing rank. This is a uniformly Σ_1^1 characterization the condition that $\rho(Q) < \rho(P)$. Equivalently, $\rho(Q) \not< \rho(P)$, if and only if there is an order preserving embedding of $R(P)$ into $R(Q)$. In the case when $R(Q)$ is well-founded, we find the embedding as we did above, and in the case when $R(Q)$ is not well-founded, we find the embedding by mapping $R(P)$ into the non-well-founded part of $R(Q)$. This gives a uniformly Π_1^1 characterization of the condition that $\rho(Q) < \rho(P)$. Consequently, ρ is a Π_1^1 -norm.

We could use a construction derived from the proof of Theorem 3.2 to show that the range of ρ is unbounded in ω_1 , and obtain a shorter proof of Corollary 3.3, that \mathbb{L} is not Borel.

Now, suppose that P is a partial ordering of ω . If P has a constrained η -extension, then the left-most path in $R(P)$ yields such an extension. Similarly, one could build a constrained η -extension of P by recursion, moving through $R(P)$ while remaining in its non-well-founded part.

When P has no constrained η -extension, $\rho(P)$ gives a measure of complexity to the reason that there is no such extension of P .

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