Optical-Flow Estimation while Preserving its Discontinuities: A Variational Approach

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Abstract. This paper describes a variational approach devised for the purpose of estimating optical flow from a sequence of images with the constraint to preserve the flow discontinuities. This problem is set as a regularization and minimization of a non quadratic functional. The *Tikhonov* quadratic regularization term usually used to recover smooth solution is replaced by a particular function of the gradient flow specifically derived to allow flow discontinuities formation in the solution. Conditions to be fulfilled by this specific regularizing term, to preserve discontinuities and insure stability of the regularization problem, are also derived. To minimize this non quadratic functional, two different methods have been investigated. The first one is an iterative scheme to solve the associated non-linear Euler-Lagrange equations. The second solution introduces dual variables so that the minimization problem becomes a quadratic or a convex functional minimization problem. Promising experimental results on synthetic and real image sequences will illustrate the capabilities of this approach.

1 Introduction

In this last decade, numerous methods have been proposed to compute 2D optical flow [8, 9, 11, 7, 4, 3, 12, 13] . Almost all these approaches use the classical constraint equation that relates the gradient of brightness to the components u and v of the local flow to estimate the optical flow. Because this problem is ill-posed, additional constraints are required. The most used one is to add a quadratic smoothness constraint as done originally by Horn and Schunk [8]. However, in order to estimate the optical flow more accurately, other constraints involving high order spatial derivatives have also been used [12]. Nevertheless, several of the proposed methods lacked robustness to the presence of occlusion, and yielded smooth optical flow. The variational approach proposed in this paper is motivated by the need to recover the optical flow while preventing the method from trying to smooth the solution across the flow discontinuities. To cope with discontinuities, several methods have been proposed [4, 13, 7]. The method presented here is inspired from a recent framework that has proven to be very useful in some image processing tasks as image restoration [14, 16, 2, 5]. It is a variational approach devised for the purpose of estimating optical flow from a sequence of images with the constraint to preserve the flow discontinuities. This problem is set as a regularization and minimization of a non quadratic functional. The *Tikhonov* quadratic regularization term usually used to recover smooth solution is replaced by a particular function of the gradient flow specifically derived to allow flow discontinuities formation in the solution.

2 Statement of the Problem

Let $I(x, y, t)$ be the grey level intensity at time t at the image point (x, y) and $u(x, y)$ and $v(x, y)$ denote the x and y components of the optical flow vector $\mathbf{v}$ at that point. Assuming that the grey level intensities are the same at time t and $t + \delta t$ leads to the well known *Optical Flow Constraint* :

$$\nabla I(x, y, t) \cdot \mathbf{v} + I_t(x, y, t) = 0 \tag{1}$$

Where ∇ is the gradient operator, and I_t denotes the temporal derivative of $I(x, y, t)$. The scalar equation (1) only permits the calculation of the components of the velocity field which is parallel to the spatial gradient ∇I (*aperture problem*). Additional constraints are therefore required to reduce the space of admissible functions. The most popular constraint is the *smoothness constraint* originally introduced by *Horn* and *Schunck* [8]. This approach combines the *Optical Flow Constraint* with the quadratic smoothing term of *Tikhonov* [15] and considers the optical flow as the global minimum of the functional:

$$F(\mathbf{v}) = \int_{\Omega} (\alpha C(\mathbf{v}) + S(\mathbf{v})) d\Omega \tag{2}$$

where α is a positive coefficient that weights the error in the image motion equation relative to the departure from smoothness. Ω denotes the image domain and the 2 terms of eq. (2) are given by :

$$\begin{cases} C(\mathbf{v}) = (I(x, y, t) \cdot \mathbf{v} + I_t(x, y, t))^2 \\ S(\mathbf{v}) = \mid \nabla u \mid^2 + \mid \nabla v \mid^2 \end{cases}$$

The minimization of this functional can be performed by writing the associated Euler-Lagrange equations :

$$\begin{cases} \nabla^2 u = \alpha I_x (I_x u + I_y v + I_t) \\ \nabla^2 v = \alpha I_y (I_x u + I_y v + I_t) \end{cases} \tag{3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplacian operator. Boundary conditions are then added and a finite difference scheme is usually used in the discrete case to solve iteratively this coupled pair of elliptic second-order partial differential equation. Unfortunately, The use of a quadratic term for $S(\mathbf{v})$ leads to a continuous and smooth solution for the optical flow i.e a solution where discontinuities are not preserved. This is not really desirable if one wants to preserve the flow discontinuities and recover the optical flow as accurately as possible because such discontinuities can be present for instance on silhouettes, where one object occludes another or between two different moving objects. The next section is devoted to presenting our approach to tackle in an efficient way this important problem.

3 A Variational Approach to Preserve Discontinuities

In order to preserve the flow discontinuities while regularizing the solution, several methods have been proposed. Some of these approaches allow to capture these discontinuities by judiciously choosing the α parameter of equation 2. Other approaches take into account gradient information of the optical flow field and recently interesting variational approaches have also been proposed to deal with this class of problem. The reader interested can refer to [4, 13, 7] where original solutions are proposed to tackle this important problem.

In the following subsection, we summarize our variational approach to optical flow estimation. This approach is inspired from the approaches developed for image restoration purpose in [14, 16]. Very recently, existence and uniqueness results have been demonstrated in [2], also for the image restoration problem. A detailed review of all these approaches can be found in [5].

3.1 The Energy Function

To cope with discontinuities, a natural way to proceed is to forbid regularizing and smoothing across such flow discontinuities. One way of taking into account these technical remarks is to consider a function $\phi(.)$ that preserves the discontinuities :

$$S(\mathbf{v}) = \phi(\mid \nabla u \mid) + \phi(\mid \nabla v \mid) \tag{4}$$

For example, a quadratic function as proposed by *Tikhonov* in [15], clearly does not preserve the discontinuities and must therefore be avoided.

Our key idea to deal with this problem is to write the Euler-Lagrange equations associated to this new functional. This leads to the following coupled pair of second order partial differential equations :

$$\begin{cases} div(\frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u) = 2\alpha(I_x u + I_y v + I_t)I_x & \\ & (x,y) \;\epsilon\, \Omega \\ div(\frac{\phi'(|\nabla v|)}{|\nabla v|}\nabla v) = 2\alpha(I_x u + I_y v + I_t)I_y & \\ & \\ \frac{\phi'(|\nabla u|)}{|\nabla u|}\frac{\partial u}{\partial n} = 0 \;\;,\;\; \frac{\phi'(|\nabla v|)}{|\nabla v|}\frac{\partial v}{\partial n} = 0 & (x,y) \epsilon\; \partial\Omega \end{cases} \tag{5}$$

where div denotes the classical divergence operator, ϕ' represents the first derivative of $\phi(s)$ with respect to the parameter s and n indicates the unit vector normal to the boundary $\partial\Omega$. Notice that the main difference with the equations 3 involving the *Tikhonov* regularization term is that the Laplacian term has been changed by the following divergence term :

$$2\nabla^2 w \longrightarrow div\left(\frac{\phi'(\mid \nabla w \mid)}{\mid \nabla w \mid}\nabla w\right) \tag{6}$$

for w being u or v. It is this replacement of the Laplacian operator by a divergence operator that will allow us to develop an anisotropic smoothing process of the flow field.

Developing and simplifying the divergence term, it can be shown that it is equal to the sum of two terms:

$$div\left(\frac{\phi'(\mid \nabla w \mid)}{\mid \nabla w \mid}\nabla w\right) = \underbrace{\frac{\phi'(\mid \nabla w \mid)}{\mid \nabla w \mid}}_{c_\xi} w_{\xi\xi} + \underbrace{\phi''(\mid \nabla w \mid)}_{c_\eta} w_{\eta\eta} \tag{7}$$

where $\phi'(s)$ and $\phi''(s)$ indicate respectively the first and second derivatives of $\phi(s)$ with respect to s, $w_{\eta\eta}$ is the second order directional derivative of w in the direction $\eta = \frac{\nabla w}{|\nabla w|}$ and $w_{\xi\xi}$ is the second order directional derivative in the direction ξ orthogonal to the gradient.

In order to regularize the solution and preserve optical flow discontinuities, one would like to smooth isotropically the optical flow field inside homogeneous regions and preserve the flow discontinuities in the inhomogeneous regions. Assuming that the function $\phi''(.)$ exists,the condition on smoothing in an isotropic way inside homogeneous regions can be achieved by imposing the following conditions on the $\phi(.)$ function:

$$\lim_{|\nabla_m w| \to 0} \frac{\phi'(\mid \nabla_m w \mid)}{\mid \nabla_m w \mid} = \lim_{|\nabla_m w| \to 0} \phi''(\mid \nabla_m w \mid) = \phi''(0) > 0 \tag{8}$$

Therefore, at the points where $\mid \nabla w \mid$ is small, the divergence term becomes

$$div\left(\frac{\phi'(\mid \nabla w \mid)}{\mid \nabla w \mid}\nabla w\right) = \phi''(0)(w_{\xi\xi} + w_{\eta\eta}) = \phi''(0)\nabla^2 w. \tag{9}$$

This case corresponds to the situation where the function $\phi(.)$ is quadratic and the smoothing is isotropic. Note that the coefficient $\phi''(0)$ is required to be positif, otherwise the regularization part will act as an inverse heat equation notably known as an instable process. Therefore, the following coupled pair of second order partial differential equations will have to be solved :

$$\begin{cases} \phi''(0)\nabla^2 u = 2\alpha(I_x u + I_y v + I_t)I_x & \\ & (x,y)\ \epsilon\,\Omega \\ \phi''(0)\nabla^2 v = 2\alpha(I_x u + I_y v + I_t)I_y & \\ & \\ \frac{\partial u}{\partial n} = 0 \quad , \quad \frac{\partial v}{\partial n} = 0 & (x,y)\epsilon\ \partial\Omega \end{cases} \tag{10}$$

In order to preserve the flow discontinuities near inhomogeneous regions presenting a strong flow gradient, one would like to smooth along the isophote (curve with constant flow) and not across them. This leads to stopping the diffusion in the gradient direction η, i.e. setting the weight $\phi''(\mid \nabla w \mid)$ to 0, while keeping a stable diffusion along the direction orthogonal ξ, i.e. setting the weight $\frac{\phi'(|\nabla w|)}{|\nabla w|}$ to some positive constant:

$$\lim_{|\nabla w| \to \infty} \phi''(\mid \nabla_m w \mid) = 0 \lim_{|\nabla w| \to \infty} (\frac{\phi'(\mid \nabla_m w \mid)}{\mid \nabla_m w \mid}) = \beta > 0 \tag{11}$$

Therefore, at the points where the flow gradient is strong, u and v will be the solutions of the following equations :

$$\begin{cases} \beta u_{\xi\xi} = 2\alpha(I_x u + I_y v + I_t) I_x & \\ & (x, y) \; \epsilon \Omega \\ \beta v_{\xi\xi} = 2\alpha(I_x u + I_y v + I_t) I_y & \\ & \\ \frac{\partial u}{\partial n} = 0 \;\; , \;\; \frac{\partial v}{\partial n} = 0 & (x, y) \epsilon \; \partial \Omega \end{cases} \tag{12}$$

Note that the positiveness of the β coefficient is also required to generate a stable smoothing process in the ξ direction.

Unfortunately, the two conditions of (11) cannot be satisfied simultaneously by a function $\phi(\mid \nabla_m Z \mid)$. However, the following conditions can be imposed in order to decrease the effects of the diffusion along the gradient more rapidly than those associated with the diffusion along the isophotes:

$$\begin{array}{l} \lim_{|\nabla_m w| \to \infty} \phi''(\mid \nabla_m w \mid) = \lim_{|\nabla_m w| \to \infty} \frac{\phi'(|\nabla_m w|)}{|\nabla_m w|} = 0 \\ \lim_{|\nabla_m w| \to \infty} \frac{\phi''(|\nabla_m w|)}{\frac{\phi'(|\nabla_m w|)}{|\nabla_m w|}} = 0 \end{array} \tag{13}$$

The conditions given by Equations (8) and (13) are those which one has to impose in order to deal with a regularization process which preserves the discontinuities. As it has been shown very recently in [2], these conditions are also sufficient to prove that the model is well posed mathematically, and the existence and uniqueness of a solution is also guaranteed by these conditions.

One can easely see that only the last two functions fulfill all the conditions mentioned above. Moreover, the convex potential functions allows to obtain uniqueness of a global minimum.

3.2 Energy Minimization

The energy function given by eq. (2) is minimized by solving the coupled pair of partial differential equations given by eq. (5). In his subsection, we present 2 different numerical schemes that have been developed for moving iteratively towards the solution.

Author	$\phi(s)$	$\frac{\phi'(s)}{s}$
Geman et Reynolds	$\frac{s^2}{1+s^2}$	$\frac{2}{(1+s^2)^2}$
Malik et Perona	$\log(1+s^2)$	$\frac{2}{(1+s^2)}$
Green	$2\log[\cosh(s)]$	$\begin{cases} 2 & s=0 \\ 2\tanh(s)/s & s\neq 0 \end{cases}$
Aubert	$2\sqrt{1+s^2}-2$	$\frac{2}{\sqrt{(1+s^2)}}$

An Implicit Scheme The first scheme that has been developed to solve the Euler-Lagrange equations of eq. (5) uses a linearized version of the discrete equations and a Gauss-Seidel relaxation method for moving iteratively towards the solution of this problem.

Implicit Scheme:

It can be shown that the divergence operator can be written as follows [10]

$$\left(div(\frac{\phi'(\mid \nabla w \mid)}{\mid \nabla w \mid}\nabla w)\right)_{i,j} \simeq \sum_{p,q\epsilon\{-1,0,1\}^2} \sigma_{w\,i+p,j+q} w_{i+p,j+q} \tag{14}$$

where the coefficients $\sigma_{w\,i+p,j+q}$ depend on the values of the 8 neighbors.

Using this result, the coupled pair of PDE of eq. (5) may be rewritten as the following non-linear system of equations

$$\begin{cases} \sum \sigma_{u\,i+p,j+q} u_{i+p,j+q} = 2\alpha\left((I_x u + I_y v + I_t)I_x\right)_{i,j} \\ \sum \sigma_{v\,i+p,j+q} v_{i+p,j+q} = 2\alpha\left((I_x u + I_y v + I_t)I_y\right)_{i,j} \\ \oplus \quad \text{Boundary conditions on the flow} \end{cases} \tag{15}$$

This large system of equations in non linear due to the fact that the divergence term includes the function to be estimated (i.e u and v). There are numerous iterative schemes for the solution of a large sparse set of equations, among them Gauss Seidel and Jacobi are the most popular. We use a linearized version of the discrete equations and we apply a Gauss-Seidel relaxation method for moving iteratively towards the solution of this problem.

Half quadratic regularization In the previous subsection, a non quadratic energy functional was minimized by solving non linear equations. In this subsection, instead of stress upon solving a non linear system, we introduce a new function which, although defined over an extended domain, has the same minimum in $\mathbf{v}$ as F and can be manipulated with linear algebraic methods.

We first start by presenting an important theorem inspired from *Geman* and *Reynolds* theorem (see [6]). A proof of this theorem can be found in [1].

Theorem 3.2.1
Let $\phi : [0, +\infty[\rightarrow [0, +\infty[$ be such that $\phi(\sqrt{x})$ is strictly concave on $]0, +\infty[$. Let L and M be defined as:

$$L = \lim_{x \to +\infty} \frac{\phi'(x)}{2x} \quad \text{and} \quad M = \lim_{x \to 0^+} \frac{\phi'(x)}{2x}$$

Then, we have the following properties:

– *there exists a strictly convex and decreasing function $\psi : [L, M] \rightarrow [\alpha, \beta]$ where*

$$\phi(x) = \liminf_{L \leq b \leq M} (bx^2 + \psi(b)) \tag{16}$$

such that:

$$\alpha = \lim_{x \to +\infty} \left(\phi(x) - x^2 \frac{\phi'(x)}{2x} \right) \ \ et \ \beta = \lim_{x \to 0_+} \phi(x)$$

– *for every fixed $x \geq 0$ the value b_x for which the minimum is reached is unique and given by:*

$$b_x = \frac{\phi'(x)}{2x} \tag{17}$$

The variable b_x is called an auxiliary variable associated to x. One can check that this theorem can be applied to all the $\phi(.)$ functions proposed for discontinuities preservations. So, introducing two dual variables b_u and b_v associated respectively to $\mid \nabla u \mid$ and $\mid \nabla v \mid$ and noting $\mathbf{b} = \begin{pmatrix} b_u \\ b_v \end{pmatrix}$, the problem becomes to minimize the resulting augmented energy with respect to $\mathbf{b}$ and $\mathbf{v}$:

$$F^*(\mathbf{v}, \mathbf{b}) = \int_\Omega C(\mathbf{v}) + \alpha S^*(\mathbf{v}, \mathbf{b}) \, d\Omega$$

where

$$S^*(\mathbf{v}, \mathbf{b}) = b_u \mid \nabla u \mid^2 + \psi(b_u) + b_v \mid \nabla v \mid^2 + \psi(b_v)$$

The advantage of considering F^* instead of F is twofold:
• for a fixed V, the functional F^* is convex in $\mathbf{b}$
• for a fixed $\mathbf{b}$, the functional F^* is convex in V

These properties are used to perform the algorithm described in table 1.
The two stages of minimization work as follows :
• calculating $\mathbf{b}^{n+1} = arg \min_{\mathbf{b}} F^*(V^n, \mathbf{b})$ amounts solving:

$$\min_{\mathbf{b} = \begin{pmatrix} b_u \\ b_v \end{pmatrix}} \int_\Omega b_u \mid \nabla u \mid^2 + \psi(b_u) + b_v \mid \nabla v \mid^2 + \psi(b_v) \, d\Omega$$

$V^0 \equiv \mathbf{0}$	
Repeat	
	$\mathbf{b}^{n+1} = arg\min_{\mathbf{b}} F^*(V^n, \mathbf{b})$
	$V^{n+1} = arg\min_V F^*(\mathbf{v}, \mathbf{b}^{n+1})$
	$n = n + 1$
Until convergence	

Table 1. Half quadratic regularization principle

The two variables b_u and b_v being independent, we can use theorem 3.2.1 and show that the minimum is reached for:

$$b_u = \frac{\phi'(\mid \nabla u \mid)}{2 \mid \nabla u \mid} \quad , b_v = \frac{\phi'(\mid \nabla v \mid)}{2 \mid \nabla v \mid} \tag{18}$$

- calculating $\mathbf{v}^{n+1} = arg\min_V F^*(\mathbf{v}, \mathbf{b}^{n+1})$ is equivalent solving:

$$\min_{u,v} \int_\Omega \alpha(\nabla I.\mathbf{v} + I_t)^2 + (b_u^F \mid \nabla u \mid^2 + b_{\mathbf{v}}^F \mid \nabla v \mid^2)\, d\Omega$$

where we noted b_w^F to recall that the auxiliary variables are now fixed. This is now easier because the functional to be minimized is quadratic and hence we only have to solve linear equations. This can be done iteratively by the classical Gauss-Seidel method. The associated Euler-Lagrange equations can be written as follows :

$$\begin{cases} div(b_u^F \nabla u) = \alpha(I_x u + I_y v + I_t) I_x \\ div(b_v^F \nabla v) = \alpha(I_x u + I_y v + I_t) I_y \\ \oplus \text{ Boundary conditions on the flow} \end{cases}$$

This algorithm converges to the unique solution of our problem. The reader interested by more details may refer to [2] where convergence results have been proved.

4 Experimental results

To validate the approach that has been presented in this paper, some experiments have been carried out using synthetic and real image sequences extracted from [3].

For each experiment (see next page), the left figure is the image for which the flow is computed, the middle one is the result obtained with a quadratic regularization term and the right figure corresponds to the result computed with the potential function proposed by *Aubert*. Parameters values are written in the caption. As expected, it appears from the figures that the discontinuities are better preserved using the specific *Aubert* function.

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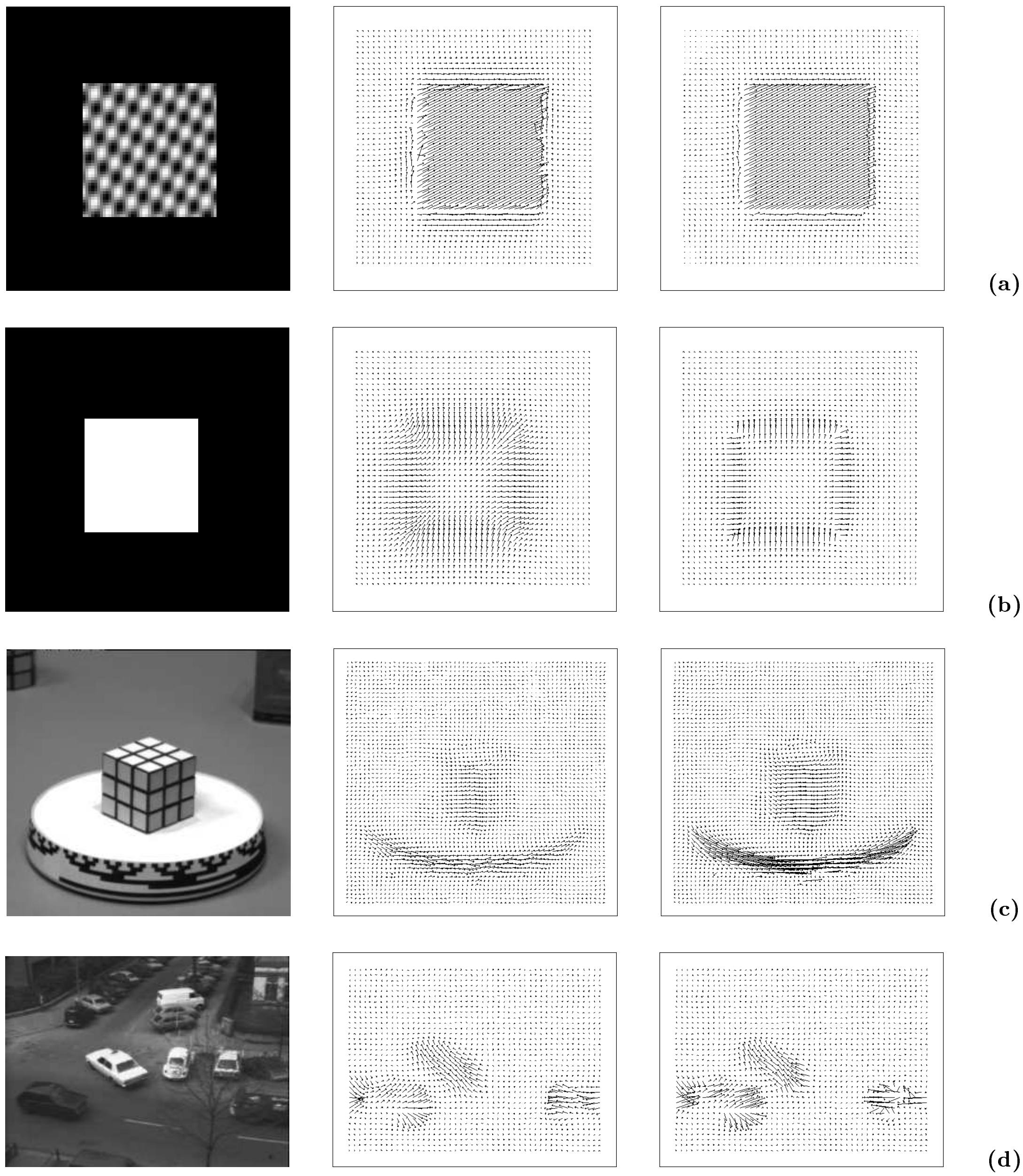

Fig. 1. **(a)** plaid pattern moving in a window: 100 iterations, $\alpha_H = 200$, $\delta = 0.5$ **(b)** translating square: 100 iterations, $\alpha_H = 10000$, $\delta = 1.0$ **(c)** Rubik's cube: 100 iterations, $\alpha_H = 10000$, $\delta = 0.5$ **(d)** Taxi data: 100 iterations, $\alpha_H = 10000$, $\delta = 0.7$